The Unitary Group in Its Strong Topology

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- $\mathcal{H}$ denotes a complex Hilbert space.

- A unitary operator on $\mathcal{H}$ is a complex linear mapping $T : \mathcal{H} \to \mathcal{H}$ which is surjective and satisfies $\langle Tf, g\rangle = \langle f, Tg\rangle$ for all $f, g \in \mathcal{H}$.

- Unitary operators are bounded and injective, hence their inverses exist and are unitary as well.

- U($\mathcal{H}$) is the group of all unitary operators on $\mathcal{H}$, the **unitary group**.

The unitary group appears in various different applications in mathematics and physics.

- Operator Theory, Mathematical Physics
- Representation Theory (of Groups, Algebras, ...)
- Geometry and Topology (action on spaces and bundles), e.g. K-Theory

In its **norm topology** $\mathsf{U}(\mathcal{H})$ is a real Banach Lie group with models in the Banach space of self-adjoint bounded operators on $\mathcal{H}$.

The **strong topology** is the topology of pointwise convergence on $\mathsf{U}(\mathcal{H})$. It is generated by the seminorms

$$p_\varphi(T) = \|T(\varphi)\| \ , T \in \mathsf{U}(\mathcal{H}), \varphi \in \mathcal{H} \, .$$

The **strong topology** τ_s is weaker than the **compact open topology** τ_c, which is in turn weaker than the **norm topology** τ_n:

$$\tau_s \subset \tau_c \subset \tau_n$$

The norm topology, however, is too strong for many purposes.

For example, for a compact group G, the left regular representation

$$\lambda : G \to \mathsf{U}(\mathsf{L}^2(G))\,, \ \text{ with } \ \lambda_g \varphi(h) := \varphi(g^{-1}h)\,,\, \varphi \in \mathsf{L}^2(G)$$

is in general not continuous with respect to the norm topology.

For this reason, the continuity condition in case of representations $\pi : G \to \mathsf{U}(\mathcal{H})$ for a topological group G is the continuity of the 'action'

$$G \times \mathcal{H} \to \mathcal{H}\,,\ (g,\varphi) \mapsto \pi(g)(\varphi)\,.$$

This joint continuity turns out to be equivalent to the continuity of $\pi : G \to \mathsf{U}(\mathcal{H})$ in the strong topology.

In this sense, the strong topology is the 'right' topology.

In research articles and books one often finds the statement: "'The unitary group is not a topological group with respect to the strong topology."'

(Bargmann, Simms, Atiyah)

As a consequence, several proofs related to the strong topology become complicated or unnatural constructions are carried through to compensate the missed property.

However,

the unitary group is a topological group!

Fact 1: U($\mathcal{H}$) *in its strong topology is a topological group.*

Proof: For the continuity of $(S, S') \mapsto S \circ S'$ at $(T, T') \in \mathsf{U}(\mathcal{H}) \times \mathsf{U}(\mathcal{H})$ one has to show:

$\forall\, \varepsilon > 0$ and $\forall\, \varphi \in \mathcal{H}$ there are neighbourhoods $\mathcal{V}$ of T and $\mathcal{V}'$ of T' such that

$$\forall\, (S, S') \in \mathcal{V} \times \mathcal{V}' \,:\, S \circ S' \in \mathcal{B}_\varphi(T \circ T', \varepsilon)\,,$$

where $\mathcal{B}_\psi(A, r) = \{U \in \mathsf{U}(\mathcal{H}) : \|U(\psi) - A(\psi)\| < r\}$ is the semiball given by ψ.

With the choices $\psi = T'\varphi$, $\mathcal{V} = \mathcal{B}_\psi(T, \varepsilon/2)$ and $\mathcal{B}_\varphi(T', \varepsilon/2)$ this condition is satisfied:

$$\begin{aligned}
\big\|S \circ S'(\varphi) - T \circ T'(\varphi)\big\| &= \big\|S \circ S'(\varphi) - T \circ S'(\varphi) + T \circ S'(\varphi) - T \circ T'(\varphi)\big\| \\
&\leq \big\|S \circ S'(\varphi) - T \circ S'(\varphi)\big\| + \big\|T \circ S'(\varphi) - T \circ T'(\varphi)\big\| \\
&\leq \|S(\psi) - T(\psi)\| + \big\|S'(\varphi) - T'(\varphi)\big\| \\
&= p_\psi(S - T) + p_\varphi(S' - T') < \varepsilon/2 + \varepsilon/2 = \varepsilon\,.
\end{aligned}$$

In order to show the continuity of $S \mapsto S^{-1}$ at T let $\varphi \in \mathcal{H}$ and $\varepsilon > 0$.

With $\psi := T^{-1}\varphi$ we conclude for every $S \in \mathcal{B}_\psi(T, \varepsilon)$:

$$\left\|S^{-1}\varphi - T^{-1}\varphi\right\| = \left\|S^{-1} \circ T\psi - \psi\right\| = \left\|S \circ S^{-1} \circ T\psi - S\psi\right\| = \|T\psi - S\psi\| < \varepsilon\,,$$

hence $S^{-1}(\mathcal{B}_\psi(T, \varepsilon)) \subset \mathcal{B}_\varphi(T, \varepsilon)$.

$\square$

Another statement in the mathematical literature is: "$\tau_s \neq \tau_c$ on U($\mathcal{H}$)."

However:

Fact 2: *The strong topology τ_s on* U($\mathcal{H}$) *coincides with the compact open topology τ_c.*

This can easily be shown by using the fact that U($\mathcal{H}$) ist equicontinuous.

The next result is also easy to show.

Fact 3: *The unitary group* $\mathrm{U}(\mathcal{H})$ *in its strong topology is metrizable in case of a separable Hilbert space* $\mathcal{H}$.

Fact 4: *The unitary group* $\mathrm{U}(\mathcal{H})$ *in its strong topology is contractible in case of* $\dim \mathcal{H} = \infty$.

This is a deep result due to Kuiper which is proven along the lines of the corresponding proof for the norm topology: The unitary group $\mathrm{U}(\mathcal{H})$ and its complexification $\mathrm{GL}(\mathcal{H})$ are contractible in the norm topology τ_n if $\mathcal{H}$ is infinite dimensional.

Question: Does there exist a manifold structure on the unitary group $\mathrm{U}(\mathcal{H})$ with its strong topology $\tau_s = \tau_c$?

Observations:

- Every s.a. operator A on $\mathcal{H}$ (bounded or unbounded) generates a one-parameter group $t \mapsto \exp itA$ of unitary operators.

- The question has to be made precise since the collection of all s.a. operators is not a vector space.

- Several considerations in physics seem to assume formally that the unitary group has such a Lie group structure.

$\diamond$

Thank You very much!

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